Indag. Mathem., N.S., 10 (3), 393–406 September 27, 1999

The spectra of some algebras of analytic mappings

by D. Garcia[1,*], M.L. Lourenço[2,**], L.A. Moraes[3,***] and O.W. Paques[4,****]

[1] *Departamento de Analisis Matematico, Universidad de Valencia, 46.100, Burjasot, Valencia, Spain, e-mail Domingo.Garcia@uv.es*

[2] *Departamento de Matemática-IME, Universidade de São Paulo, CP 66281, CEP 05389-970, São Paulo, Brazil, e-mail. mllouren@ime.usp.br*

[3] *Instituto de Matemática, Universidade Federal do Rio de Janeiro, CP 68530, CEP 21945-970, Rio de Janeiro, Brazil, e-mail. luiza@dmm.im.ufrj.br*

[4] *Departamento de Matemática, IMECC-UNICAMP, CEP. 13081-970, Campinas, SP, Brazil, e-mail otilia@ime unicamp br*

Communicated by Prof. I Gohberg at the meeting of June 22, 1998

ABSTRACT

Let E be a Banach space with the approximation property and let F be a Banach algebra with identity. We study the spectrum of the algebra $\mathcal{H}_b(E, F)$ of all holomorphic mappings $f : E \to F$ that are bounded on the bounded subsets of E.

1 INTRODUCTION

If X is a topological algebra with identity, let $\mathcal{M}(X)$ be the spectrum of X i.e., $\mathcal{M}(X)$ consists of all non-zero continuous complex valued homomorphisms ϕ from X to $\mathbb{C}$. Note that, for every set A, $A \times \mathcal{M}(\mathbb{C})$ is isomorphic to A.

Throughout E will be a complex Banach space with open unit ball B_E and F will be a complex Banach algebra with identity 1_F except when indicated otherwise. The bidual E'' of E endowed with the strong topology β will be also denoted by E''. As usual, w and w^* will denote, respectively, the weak topology $\sigma(E, E')$ in E and the weak-star topology $\sigma(E'', E')$ in E''. If f is a mapping with values in a Banach space Y and B is contained in the domain of f, $\|f\|_B$ denotes the $\sup_{x \in B} \|f(x)\|$.

As usual, the space of all continuous n-homogeneous polynomials from E into F is denoted by $\mathcal{P}(^nE, F)$, the space $\sum_{n=0}^{\infty} \mathcal{P}(^nE, F)$ is denoted by $\mathcal{P}(E, F)$

---

* Supported in part by DGICYT pr. 91-0326 and by grant 92-162.

** Supported in part by FAPESP, Brazil and the exchange Ccint-USP-UV.

*** Supported in part by CNP_q, Brazil, Research Grant 300016/82-4 and by FINEP.

**** Supported in part by CNP_q, Brazil, Research Grant 300686/89-7.

and the space of all holomorphic mappings from E into F that are bounded on the bounded subsets of E is denoted by $\mathcal{H}_b(E,F)$. Let $\mathcal{P}_f({}^nE) := \operatorname{span}\{\phi^n : \phi \in E'\}$, $\mathcal{P}_f({}^nE,F) := \mathcal{P}_f({}^nE) \otimes F$ and $\mathcal{P}_f(E,F) := \sum_{n=0}^{\infty} \mathcal{P}_f({}^nE,F)$. We denote by $\mathcal{P}_K({}^nE,F)$ the space of the elements $P \in \mathcal{P}({}^nE,F)$ that are compact (i.e., $\overline{P(B)}$ is relatively compact for every bounded subset B of E). We consider $\mathcal{H}_b(E,F)$, and all the spaces of polynomials defined above, endowed with the topology τ_b of uniform convergence on the bounded subsets of E. As usual, we will always omit F in the notation in case $F = \mathbb{C}$. So, for instance, we will write $\mathcal{H}_b(E)$ for $\mathcal{H}_b(E,\mathbb{C})$. It is known that $\mathcal{H}_b(E,F)$ is a Fréchet algebra with identity.

Aron, Cole and Gamelin showed that if E' has the approximation property then $\mathcal{M}(\mathcal{H}_b(E))$ coincides with E'' (as a point set) if and only if $\mathcal{P}_f(E)$ is dense in $\mathcal{H}_b(E)$ (cf. Theorem 7.2 of [6]). Motivated by this result, we consider the problem of determining the spectrum of $\mathcal{H}_b(E,F)$ when F is an arbitrary Banach algebra with identity. Our general approach is to obtain the spectrum of the algebra $\mathcal{H}_{bc}(E,F)$, where $\mathcal{H}_{bc}(E,F)$ is the closure of $\mathcal{P}_f(E,F)$ in $\mathcal{H}_b(E,F)$. This is the purpose of Section 2. According to [18] every $f \in \mathcal{H}_{bc}(E,F)$ has a unique extension $\tilde{f} : E'' \to F$ whose restriction to the bounded subsets of E'' is w^*-continuous. Thus we are able to define δ_z, for all $z \in E''$, to be the evaluation at z of the above cited extension and we show that, given any $(z,\varphi) \in E'' \times \mathcal{M}(F)$, the mapping $\delta_z \otimes \varphi$ defined by $\delta_z \otimes \varphi(f) := \varphi(\delta_z(\tilde{f}))$ for all $f \in \mathcal{H}_{bc}(E,F)$ is in the spectrum of $\mathcal{H}_{bc}(E,F)$. Finally we establish a bijection $\delta_z \otimes \varphi \to (z,\varphi)$ from $\mathcal{M}(\mathcal{H}_{bc}(E,F))$ onto $E'' \times \mathcal{M}(F)$ (cf. Theorem 2.1).

In Section 3 we consider the space $\mathcal{H}_{bK}(E,F)$ of the elements of $\mathcal{H}_b(E,F)$ that are compact and show that if E' has the approximation property then $\mathcal{M}(\mathcal{H}_{bK}(E,F))$ coincides with $E'' \times \mathcal{M}(F)$ whenever F has the approximation property if and only if $\mathcal{P}_f(E)$ is dense in $\mathcal{P}(E)$ (cf. Corollary 3.3).

The purpose of Section 4 is to get a type of corona theorem for $\mathcal{H}_{bc}(E,F)$. For discussions of the classical corona problem we refer to [17]. We start establishing that whenever we consider a spectrum as a topological space we will be considering the spectrum with the Gelfand topology. We show that the bijection π from $\mathcal{M}(\mathcal{H}_{bc}(E,F))$ onto $E'' \times \mathcal{M}(F)$ established in Section 2 is continuous when we consider E'' endowed with the weak-star topology (cf. Theorem 4.2). We get a type of corona theorem when we show that $E \times \mathcal{M}(F)$ is dense in $\mathcal{M}(\mathcal{H}_{bc}(E,F))$ (cf. Theorem 4.1).

Section 5 is devoted to study the spectrum of $\mathcal{H}_b(E,F)$. We define a mapping $\hat{\delta} : E'' \times \mathcal{M}(F) \to \mathcal{M}(\mathcal{H}_b(E,F))$ and show that if E' has the approximation property and F is a uniform Banach algebra with identity then $\hat{\delta}$ defines a bijection from $E'' \times \mathcal{M}(F)$ onto $\mathcal{M}(\mathcal{H}_b(E,F))$ if and only if $\mathcal{P}_f(E,F)$ is dense in $\mathcal{P}(E,F)$ (cf. Theorem 5.2). This result gives a generalization of Theorem 7.2 of [6]. We also get a type of corona theorem in case of $\mathcal{M}(\mathcal{H}_b(E,F))$ (cf. Theorem 5.3).

Finally, in Section 6 we consider the Banach algebra $\mathcal{A}_{wu}(B_E;F)$ of all $f : \overline{B}_E \to F$ that are holomorphic on B_E and uniformly weakly continuous on $\overline{B}_E$. We prove that, if E' has the approximation property, the spectrum of

$\mathcal{A}_{wu}(B_E;F)$ is homeomorphic to $(\overline{B}_{E''},w^*)\times\mathcal{M}(F)$ (cf. Theorem 6.4). Consequently, we have the corona theorem for $\mathcal{A}_{wu}(B_E;F)$.

For background on holomorphic functions in infinite dimensions and for Banach and Fréchet algebras we refer to [12], [20], [14].

2 THE ALGEBRAS $\mathcal{H}_{bc}(E,F)$ AND $\mathcal{H}_{wu}(E,F)$

Let $\mathcal{P}_c(^nE,F)$ be the closure of $\mathcal{P}_f(^nE,F)$ in $\mathcal{P}(^nE,F)$ and let $\mathcal{H}_{bc}(E,F)$ be the space of all f in $\mathcal{H}_{bc}(E,F)$ such that $\hat{d}^nf(x)$ belongs to $\mathcal{P}_c(^nE,F)$ for every $n\in IN$ and for every $x\in E$. The spaces $\mathcal{P}_c(^nE,F)$ and $\mathcal{H}_{bc}(E,F)$ have been studied by Gupta in [15]. From now on, $\mathcal{H}_{bc}(E,F)$ will denote $(\mathcal{H}_{bc}(E,F),\tau_b)$. It is easy to verify that $\mathcal{H}_{bc}(E,F)$ is a Fréchet algebra and $\mathcal{P}_f(E,F)$ is a dense subset of $\mathcal{H}_{bc}(E,F)$.

We denote by $\mathcal{H}_{wu}(E,F)$ the space of all holomorphic mappings from E into F that, when restricted to any bounded subset of E, are uniformly weakly continuous and we denote by $\mathcal{H}_{w^*u}(E'',F)$ the space of all holomorphic mappings from E'' into F that, when restricted to any bounded subset of E'', are (uniformly) w^*-continuous. Let $\mathcal{P}_{wu}(^nE,F)=\mathcal{P}(^nE,F)\cap\mathcal{H}_{wu}(E,F)$ and $\mathcal{P}_{w^*u}(^nE'',F)=\mathcal{P}(^nE'',F)\cap\mathcal{H}_{w^*u}(E'',F)$. The spaces $\mathcal{P}_{wu}(^nE,F)$ and $\mathcal{H}_{wu}(E,F)$ have been extensively studied by Aron in [3], by Aron and Prolla in [9] and by Aron, Hervés and Valdivia in [8]. It is known that $\mathcal{H}_{bc}(E,F)\subset\mathcal{H}_{wu}(E,F)\subset\mathcal{H}_b(E,F)$ for every Banach spaces E and F, and it is clear that $\mathcal{H}_{w^*u}(E'',F)\subset\mathcal{H}_b(E'',F)$. It is also known that E' has the approximation property if and only if $\mathcal{P}_{wu}(^nE,F)=\mathcal{P}_c(^nE,F)$ for every $n\in IN$ and for all Banach spaces F (cf. Proposition 2.7 of [9]). By using Proposition 1.5 of [3] and the above results we can prove that E' has the approximation property if and only if $\mathcal{H}_{bc}(E,F)=\mathcal{H}_{wu}(E,F)$ for all Banach spaces F. We consider $\mathcal{H}_{wu}(E,F)$ endowed with the topology τ_b and it is easy to show that $\mathcal{H}_{wu}(E,F)$ is a Fréchet algebra. We recall the following result, which will be useful here:

Theorem A. *Let E and F be Banach spaces. There is a unique isomorphism $T:\mathcal{H}_{wu}(E,F)\to\mathcal{H}_{w^*u}(E'',F)$ such that $(Tf)|_E=f$ for all $f\in\mathcal{H}_{wu}(E,F)$.*

Proof. See Theorem 8 and Remark 9 of [18]. □

For each $f\in\mathcal{H}_{wu}(E,F)$ we will denote by $\tilde{f}$ the unique extension of f that belongs to $H_{w^*u}(E'',F)$. For every $z\in E''$, let $\delta_z:H_{wu}(E,F)\to F$ be defined by $\delta_z(f):=\tilde{f}(z)$ for all $f\in H_{wu}(E,F)$. We recall that, in this paper, F is a Banach algebra with identity 1_F and so, given $f:E\to F$ and $g:E\to F$, it makes sense to consider the product $(fg)(x):=f(x)\cdot g(x)$ for every $x\in E$. By the uniqueness of the extension it follows that $\widetilde{f\cdot g}=\tilde{f}\cdot\tilde{g}$ for every $f,g\in H_{wu}(E,F)$. For each mapping $f:E\to\mathbb{C}$ and $b\in F$ we set $f\otimes b(x):=f(x)\cdot b$ for all $x\in E$. Let 1 denote the constant mapping that associates to every $x\in E$ the element $1\in\mathbb{C}$. It is clear that $1\otimes 1_F$ is the constant mapping which associates to each $x\in E$ the element $1_F\in F$.

Theorem 2.1. (1) *Given $\phi \in \mathcal{M}(\mathcal{H}_{bc}(E,F))$, let $\varphi(f) := \phi(f \otimes 1_F)$ for every $f \in \mathcal{H}_{bc}(E)$ and $\psi(b) := \phi(1 \otimes b)$ for every $b \in F$. Then, $\varphi|_{E'} \in E''$ and $\psi \in \mathcal{M}(F)$.*

(2) *With the notation defined in* (1), *the mapping*

$$\pi : \mathcal{M}(\mathcal{H}_{bc}(E,F)) \to E'' \times \mathcal{M}(F)$$

defined by $\pi(\phi) := (\varphi|E', \psi)$ for every $\phi \in \mathcal{M}(\mathcal{H}_{bc}(E,F))$ is injective and onto.

Proof. (1) All we have to show is that $\psi \neq 0$. As $f = f \cdot (1 \otimes 1_F)$ we have $\phi(f) = \phi(f) \cdot \phi(1 \otimes 1_F) = \phi(f) \cdot \psi(1_F)$ for every $f \in \mathcal{H}_{bc}(E,F)$. Thus, by choosing $f \in \mathcal{H}_{bc}(E,F)$ such that $\phi(f) \neq 0$ we get $\psi(1_F) \neq 0$.

(2) Given $\phi_1, \phi_2 \in \mathcal{M}(\mathcal{H}_{bc}(E,F))$ we set $\varphi_i(f) = \phi_i(f \otimes 1_F)$ for all $f \in \mathcal{H}_{bc}(E)$ and $\psi_i(b) = \phi_i(1 \otimes b)$ for all $b \in F$ $(i = 1,2)$. It is clear that $\pi(\phi_i) = \pi(\phi_2)$ implies $\varphi_1|\mathcal{P}_f(^nE) = \varphi_2|\mathcal{P}_f(^nE)$ for all $n \in IN$ and $\psi_1 = \psi_2$. Given $P \in \mathcal{P}_f(^nE)$ and $b \in F$ arbitrary, it is clear that $P \otimes b = (P \otimes 1_F) \cdot (1 \otimes b)$ and so, since $\phi_i \in \mathcal{M}(\mathcal{H}_{bc}(E,F))$ $(i = 1,2)$ we have $\phi_1(P \otimes b) = \phi_1(P \otimes 1_F) \cdot \phi_1(1 \otimes b) = \phi_2(P \otimes 1_F) \cdot \phi_2(1 \otimes b) = \phi_2(P \otimes b)$ and consequently $\phi_1|\mathcal{P}_f(E,F) = \phi_2|\mathcal{P}_f(E,F)$. Therefore π is injective, since $\phi_i (i = 1,2)$ is continuous and $\mathcal{P}_f(E,F)$ is dense in $(\mathcal{H}_{bc}(E,F), \tau_b)$.

It remains to show that π is onto. Let $(z, \lambda) \in E'' \times \mathcal{M}(F)$ be arbitrary. By Theorem A, for every $f \in \mathcal{H}_{bc}(E,F) \subset \mathcal{H}_{wu}(E,F)$ there exists a unique $\tilde{f} \in \mathcal{H}_{w^*u}(E'',F)$ such that $\tilde{f}|_E = f$. From the unicity of the extension we infer that $\tilde{u}(z) = z(u)$ for every $u \in E'$ and for every $z \in E''$, $\tilde{h} \otimes 1_F = \widetilde{h \otimes 1_F}$ for every $h \in E'$ and $\widetilde{f \cdot g} = \tilde{f} \cdot \tilde{g}$ for every $f, g \in \mathcal{H}_{bc}(E,F)$. So we can define a mapping $\delta_z \otimes \lambda : \mathcal{H}_{bc}(E,F) \to \mathbb{C}$ by setting $(\delta_z \otimes \lambda)(f) := \lambda(\tilde{f}(z))$ and it is easy to verify that $\delta_z \otimes \lambda$ is a homomorphism. For every $f \in E'$ we have $(\delta_z \otimes \lambda)(f \otimes 1_F) = \lambda(\widetilde{f \otimes 1_F}(z)) = \lambda(\tilde{f}(z) \cdot 1_F) = \tilde{f}(z) = z(f)$ and for every $b \in F$ we have $(\delta_z \otimes \lambda)(1 \otimes b) = \lambda(\widetilde{1 \otimes b}(z)) = \lambda(b)$. Since $\lambda \in \mathcal{M}(F)$, $\lambda(b) \neq 0$ for some $b \in F$ and therefore $(\delta_z \otimes \lambda)(1 \otimes b) = \lambda(b) \neq 0$. This completes the proof. $\square$

Corollary 2.2. *If E' has the approximation property we have*

$$\mathcal{M}(\mathcal{H}_{wu}(E,F)) = E'' \times \mathcal{M}(F) \quad (\text{as a point set}).$$

Proof. It is enough to remember that $\mathcal{H}_{wu}(E,F) = \mathcal{H}_{bc}(E,F)$ if E' has the approximation property and the result follows from Theorem 2.1. $\square$

Corollary 2.3. *If $\mathcal{P}_f(E,F)$ is dense in $\mathcal{P}(E,F)$, then $\mathcal{M}(\mathcal{H}_b(E,F))$ coincides with $E'' \times \mathcal{M}(F)$ (as a point set).*

Proof. If $\mathcal{P}_f(E,F)$ is dense in $\mathcal{P}(E,F)$, it follows from the Cauchy inequalities that $\mathcal{P}_f(^nE,F)$ is dense in $\mathcal{P}(^nE,F)$ $\forall n \in IN$ i.e., $\mathcal{P}(^nE,F) = \mathcal{P}_c(^nE,F)$ $\forall n \in IN$. Therefore, $\mathcal{H}_b(E,F) = \mathcal{H}_{bc}(E,F)$. $\square$

Corollary 2.4. *If $c_0 \not\subset F$ we have $\mathcal{M}(H_b(c_0,F)) = l_\infty \times \mathcal{M}(F)$ (as a point set).*

Proof. Since c_0 is isomorphic to $C(IN \cup \{\infty\})$ and $IN \cup \{\infty\}$ is a dispersed compact Hausdorff space we have that $\mathcal{P}_f({}^n c_0, F)$ is dense in $\mathcal{P}({}^n c_0, F)$ for every $n \in IN$ and for every F such that $c_0 \not\subset F$ (cf. example below). Consequently, $\mathcal{H}_b(c_0, F) = \mathcal{H}_{bc}(c_0, F)$ for every F such that $c_0 \not\subset F$. $\square$

Next we are going to give some examples of Banach spaces E and F such that $\mathcal{P}_f(E, F)$ is dense in $\mathcal{P}(E, F)$. We also show that F can be always considered as a Banach algebra with identity (see remark after the examples). So, we are going to get examples of Banach spaces E and F such that $\mathcal{M}(H_b(E, F))$ coincides with $E'' \times \mathcal{M}(F)$ as a point set.

Example 2.1. Let $E = \mathcal{C}(X)$ where X is a dispersed compact Hausdorff space. It is known that $\mathcal{P}_K({}^n\mathcal{C}(X), F) = \mathcal{P}({}^n\mathcal{C}(X), F)$ for every $n \in IN$ and for every Banach space F such that $c_0 \not\subset F$ if and only if X is a dispersed compact Hausdorff space (cf. [21]). By corollary in §2 of [4] $\mathcal{P}_f({}^n\mathcal{C}(X))$ is dense in $\mathcal{P}({}^n\mathcal{C}(X))$ for every $n \in IN$ and, since $\overline{\mathcal{P}_f({}^n\mathcal{C}(X))} \subset \mathcal{P}_{wu}({}^n\mathcal{C}(X))$, we have $\mathcal{P}_{wu}({}^n\mathcal{C}(X)) = \mathcal{P}({}^n\mathcal{C}(X))$. By Theorem 2.9 of [8], for every Banach spaces E and F the space $\mathcal{P}_w({}^nE, F)$ of elements of $\mathcal{P}({}^nE, F)$ that are weakly continuous on the bounded subsets of E coincides with $\mathcal{P}_{wu}({}^nE, F)$. So we have $\mathcal{P}_w({}^n\mathcal{C}(X)) = \mathcal{P}({}^n\mathcal{C}(X))$ and as a consequence we get $\mathcal{P}_K({}^n\mathcal{C}(X), F) \subset \mathcal{P}_w({}^n\mathcal{C}(X), F)$ for every $n \in IN$ and for every Banach space F (cf. [19]). Now it is enough to remember that $\mathcal{C}(X)'$ has the approximation property to get the density of $\mathcal{P}_f({}^n\mathcal{C}(X), F)$ in $\mathcal{P}({}^n\mathcal{C}(X), F)$ for every $n \in IN$ and for every Banach space F such that $c_0 \not\subset F$. Let

$$l_1 = \{(x_n)_{n=-\infty}^{+\infty} \subset \mathbb{C} : \sum_{n=-\infty}^{+\infty} |x_n| < \infty\} \text{ and let}$$

$$l_1^+ = \{(x_n)_{n=0}^{+\infty} \subset \mathbb{C} : \sum_{n=0}^{+\infty} |x_n| < \infty\}$$

l_1 and l_1^+ with the usual norm. It is known that l_1 endowed with the convolution product is a Banach algebra with identity $e = (y_n)_{n=-\infty}^{n=\infty}$ where $y_n = 0$ for every $n \neq 0$ and $y_0 = 1$. So l_1^+ is also a Banach algebra with identity and both do not contain c_0. It is also known that the spectrum of l_1 can be identified with the unit circle of $\mathbb{C}$ and the spectrum of l_1^+ can be identified with the closed unit disc of $\mathbb{C}$ (for details cf. 1.4.13 and 1.4.15 of [22]).

Example 2.2. Let T be the Tsirelson space (cf. [23]). By Theorem 2.3. (1) of [2], we have that $\mathcal{P}({}^nT, F)$ coincides with the space of the weak-to-norm sequentially continuous polynomials if F has positive rank. We recall that a Banach space has positive rank if there exists $\alpha \in (0, 1)$ such that every sequence (x_n) satisfying $\|\sum_{n\in B} x_n\| \leq c|B|^\alpha$ for $c \geq 0$ and for all finite $B \subset IN$ (where $|B|$ is the number of elements of B) converges with respect to the norm.

Since T is reflexive, using Theorem 3 of [16] we get $\mathcal{P}({}^nT, F) = \mathcal{P}_w({}^nT, F)$ $(= \mathcal{P}_{wu}({}^nT, F))$. Now it is enough to remark that the dual of T has the approximation property. We note that T' and l_p $(1 < p < \infty)$ are examples of Banach spaces with positive rank (cf. [2]).

Example 2.3. Let T_J' be the James space modelled on T (cf. [7]). From Theorem 2.4 (4) of [2] we infer that $\mathcal{P}(^nT_J', F)$ is the space of the n-homogeneous polynomials from T_J' into F that are weak-to-norm sequentially continuous for every Banach space F of positive rank. Since $l_1 \not\subset T_J'$ we have $\mathcal{P}_w(^nT_J', F) = \mathcal{P}(^nT_J', F)$ by Theorem 3 of [16]. Now it is enough to remark that the dual of T_J' has the approximation property.

Remark. Given an arbitrary Banach space $(E, \| \ \|)$ we can always define a product $\odot$ on E in order that $(E, +, \odot)$ is an algebra with identity. We can also define in E a norm $||| \ |||$ that is equivalent to the original norm $\| \ \|$ and such that $(E, ||| \ |||)$ with the above operations is a Banach algebra (with identity). We are grateful to Aron who showed us the following proof of this statement. Consider a closed hyperplane F of E, take $e \in E \setminus F$ such that $\|e\| = 1$ and define $\odot$ in the following way: if $u = ae + x$ and $v = be + y$ (where $a, b \in \mathbb{C}$ and $x, y \in F$) $u \odot v := abe + (bx + ay)$. It is easy to verify that $(E, +, \odot)$ is an algebra with identity e. Now, if we define $|||u||| := |a| + \|x\|$ for $u = ae + x$ ($a \in \mathbb{C}$ and $x \in F$) we get an equivalent norm on E and $(E, ||| \ |||)$ endowed with $+$ and $\odot$ is a Banach algebra with identity e.

3. THE ALGEBRA $\mathcal{H}_{bK}(E, F)$

Let $\mathcal{H}_{bK}(E, F)$ be the space of all $f \in \mathcal{H}_b(E, F)$ such that there is a 0-neighbourhood V_0 in E such that $f(V_0)$ is relatively compact in F. Let $\mathcal{P}_K(^nE, F) = \mathcal{P}(^nE, F) \cap \mathcal{H}_{bK}(E, F)$. By Proposition 3.4 of [10] we have that $\mathcal{H}_{bK}(E, F)$ is the space of all $f \in \mathcal{H}_b(E, F)$ such that $\frac{d^nf(0)}{n!} \in \mathcal{P}_K(^nE, F)$ for every $n \in IN$. It is easy to verify that $(\mathcal{H}_{bK}(E, F), \tau_b)$ is a Fréchet algebra. We denote this algebra also by $\mathcal{H}_{bK}(E, F)$. It is easy to show that $\mathcal{H}_b(E) \otimes F$ is dense in $\mathcal{H}_{bK}(E, F)$ if F has the approximation property. Indeed, by Proposition 3.5 of [10] $K = \overline{f(B_E)}$ is a compact subset of F if $f \in \mathcal{H}_{bK}(E, F)$; as F has the approximation property, given $\epsilon > 0$ there exist $\varphi_1, \dots, \varphi_k \in F'$ and $b_1, \dots, b_k \in F$ such that $\| \sum_{j=1}^k \varphi_j(x) \cdot b_j - x\| < \epsilon$ for all $x \in K$. So, if $T(x) = \sum_{j=1}^k \varphi_j(x) \cdot b_j$ it is clear that $T \circ f \in \mathcal{H}_b(E, F)$ and, as $f(x) \in K$ for all $x \in B_E$, $\|T \circ f(x) - f(x)\| < \epsilon$ for all $x \in B_E$. Finally we note that $T \circ f(x) = \sum_{j=1}^k \varphi_j(f(x)) \cdot b_j$ and $\varphi_j \circ f \in \mathcal{H}_b(E)$ for $j = 1, \cdots, k$.

By using this we can prove:

Proposition 3.1. *If F has the approximation property, then there exists a bijection from $\mathcal{M}(\mathcal{H}_{bK}(E, F))$ onto $\mathcal{M}(\mathcal{H}_b(E)) \times \mathcal{M}(F)$.*

Proof. Given $\phi \in \mathcal{M}(\mathcal{H}_{bK}(E, F))$, let $\varphi(f) := \phi(f \otimes 1_F)$ for every $f \in \mathcal{H}_b(E)$ and $\psi(b) := \phi(1 \otimes b)$ for every $b \in F$. It is easy to show that $\psi \in \mathcal{M}(F)$ and that φ is a continuous homomorphism in $\mathcal{H}_b(E)$. Since $\mathcal{H}_b(E) \otimes F$ is dense in $\mathcal{H}_{bK}(E, F)$ and ϕ is a non zero continuous homomorphism in $\mathcal{H}_{bK}(E, F)$ we get that $\varphi \neq 0$ and so $\varphi \in \mathcal{M}(\mathcal{H}_b(E))$. Now we define $\pi_1 : \mathcal{M}(\mathcal{H}_{bK}(E, F)) \to \mathcal{M}(\mathcal{H}_b(E)) \times \mathcal{M}(F)$ by $\pi_1(\phi) = (\varphi, \psi)$ for every $\phi \in \mathcal{M}(\mathcal{H}_{bK}(E, F))$. We claim that π_1 is a bijection. Indeed, given $\phi_1, \phi_2 \in \mathcal{M}(\mathcal{H}_{bK}(E,F))$ such that $\pi_1(\phi_1) = \pi_1(\phi_2)$ it is clear that $\varphi_1 = \varphi_2$ and $\psi_1 = \psi_2$ (where, for $i = 1, 2$, φ_i and ψ_i are the homomorphisms associated to ϕ_i). As F has the approximation property, given $f \in \mathcal{H}_{bK}(E,F)$ there exists a net in $\mathcal{H}_b(E) \otimes F$ that converges to f in $\mathcal{H}_{bK}(E,F)$. Since ϕ_i is continuous ($i = 1,2$) and $\phi_i(h \otimes b) = \varphi_i(h) \cdot \psi_i(b)$ for all $h \in \mathcal{H}_b(E)$ and $b \in F$, ($i = 1,2$) it is easy to see that $\phi_1(f) = \phi_2(f)$ and so π_1 is injective.

It remains to show that π_1 is onto. Let $(\varphi, \psi) \in \mathcal{M}(\mathcal{H}_b(E)) \times \mathcal{M}(F)$. If for each $f \in \mathcal{H}_{bK}(E,F)$ we define $T_f : F'_\beta \to \mathcal{H}_b(E)$ by $T_f(\eta)(x) := \eta(f(x))$ for all $\eta \in F'$ and $x \in E$, it is easy to see that $T_f(\eta) \in \mathcal{H}_b(E)$ for all $\eta \in F'$ and T_f is a continuous linear mapping. Let $\phi : \mathcal{H}_{bK}(E,F) \to \mathbb{C}$ defined by $\phi(f) := \varphi(T_f(\psi))$ all $f \in \mathcal{H}_{bK}(E,F)$. As $\varphi \in \mathcal{M}(\mathcal{H}_b(E))$ we have $\varphi(f) \neq 0$ for some $f \in \mathcal{H}_b(E)$ and therefore $\phi(f \otimes 1_F) \neq 0$. It is easy to verify that ϕ is a continuous homomorphism.

Finally, since $T_{f \otimes 1_F}(\psi) \equiv f$ for all $f \in \mathcal{H}_b(E)$ and $T_{1 \otimes b}(\psi) = \psi(b)$ for all $b \in F$, $\pi_1(\phi) = (\varphi, \psi)$ holds. □

Proposition 3.2. *If F has the approximation property and $\mathcal{P}_f(E)$ is dense in $\mathcal{P}(E)$, then $\mathcal{H}_{bK}(E,F) = \mathcal{H}_{bc}(E,F)$ and consequently, $\mathcal{M}(\mathcal{H}_{bK}(E,F)) = E'' \times \mathcal{M}(F)$ (as a point set).*

Proof.
Let $f \in \mathcal{H}_{bK}(E,F)$. As $\mathcal{H}_b(E) \otimes F$ is dense in $\mathcal{H}_{bK}(E,F)$, for every bounded subset B of E and $\epsilon > 0$ there exist $g_i \in \mathcal{H}_b(E)$ and $b_i \in F (i = 1, \dots, n)$ such that $\sup_{x \in B} \|f(x) - \sum_{i=1}^n (g_i \otimes b_i)(x)\| < \frac{\epsilon}{2}$. So, by using the density of $\mathcal{P}_f(E)$ in $\mathcal{P}(E)$ we get $P_i \in \mathcal{P}_f(E) (i = 1, \dots n)$ such that $\|f(x) - \sum_{i=1}^n (P_i \otimes b_i)(x)\| < \epsilon$ for all $x \in B$. Now, the Cauchy inequalities give $\frac{d^n f(0)}{n!} \in \mathcal{P}_c({}^n E, F)$ for every $n \in IN$ and by Proposition 5, Chapter II of [15] we have $f \in \mathcal{H}_{bc}(E,F)$. The other inclusion is always true. □

Corollary 3.3. *If E' has the approximation property, the following are equivalent:*

(1) *$\mathcal{M}(\mathcal{H}_{bK}(E,F)) = E'' \times \mathcal{M}(F)$ (as a point set) whenever F has the approximation property.*

(2) *$\mathcal{P}_f(E)$ is dense in $\mathcal{P}(E)$.*

Proof. By Proposition 3.1 in case $F = \mathbb{C}$ and Proposition 7.2 of [6] we get (1) implies (2). The converse is a consequence of Proposition 3.2. □

4 THE CORONA THEOREM FOR $\mathcal{H}_{bc}(E,F)$

Let A be any Fréchet algebra. As $\mathcal{M}(A) \subset A'$, it makes sense to consider in $\mathcal{M}(A)$ the weak-star topology of A' restricted to $\mathcal{M}(A)$. This topology is usually called the Gelfand topology (or the A-convergence topology) and will be denoted here by τ_G. So, given a net (ϕ_α) of elements in $\mathcal{M}(A)$, we say that ϕ_α converges to $\phi \in \mathcal{M}(A)$ in the Gelfand topology (and denote $\phi_\alpha \xrightarrow{\tau_G} \phi$) if and only if $\phi_\alpha(f) \to \phi(f)$ for every $f \in A$. Unless we say the contrary, we will be always considering $\mathcal{M}(A)$ endowed with the Gelfand topology. It is known that if F is a Banach algebra then $\mathcal{M}(F) \subset B_{F'}$ is a compact Hausdorff space.

By Theorem 2.1, π^{-1} is a bijection between the sets $E'' \times \mathcal{M}(F)$ and $\mathcal{M}(\mathcal{H}_{bc}(E,F))$. Now we are going to prove that $\pi^{-1}(E \times \mathcal{M}(F))$ is dense in $\mathcal{M}(\mathcal{H}_{bc}(E,F))$, i.e., given $\phi \in \mathcal{M}(\mathcal{H}_{bc}(E,F)) = \pi^{-1}(E'' \times \mathcal{M}(F))$ there exist $(z_\alpha)_{\alpha \in I} \subset E$ and $(\lambda_\alpha)_{\alpha \in I} \subset \mathcal{M}(F)$ such that $\pi^{-1}(z_\alpha, \lambda_\alpha)(f) \to \phi(f)$ for every $f \in \mathcal{H}_{bc}(E,F)$.

Theorem 4.1. *If $\tilde{\delta} : E'' \times \mathcal{M}(F) \to \mathcal{M}(\mathcal{H}_{bc}(E,F))$ is the mapping defined by $\tilde{\delta}(z,\lambda) := \delta_z \otimes \lambda$ for every $(z,\lambda) \in E'' \times \mathcal{M}(F)$ (where $\delta_z \otimes \lambda$ is defined as in the proof of Theorem* 2.1*), then $\tilde{\delta}(E \times \mathcal{M}(F))$ is dense in $\mathcal{M}(\mathcal{H}_{bc}(E,F))$.*

Proof. By Theorem 2.1, given $\phi \in \mathcal{M}(\mathcal{H}_{bc}(E,F))$ there exists a unique $(z,\lambda) \in E'' \times \mathcal{M}(F)$ such that $\phi = \delta_z \otimes \lambda$ via π and we have $\lambda(b) = \phi(1 \otimes b)$ for every $b \in F$ and $\phi(f) = \lambda(\tilde{f}(z))$ for every $f \in \mathcal{H}_{bc}(E,F)$ (where $\tilde{f} \in \mathcal{H}_{w^*u}(E'',F)$ is given by Theorem A). Now, given $z \in E''$ there exists $(z_\alpha)_{\alpha \in I} \subset E$ such that $\|z_\alpha\| \leq \|z\|$ for every $\alpha \in I$ and $z_\alpha \xrightarrow{w^*} z$. Since $\tilde{f}$ is w^*-continuous on the bounded subsets of E'', we have $\tilde{f}(z_\alpha) \to \tilde{f}(z)$ and therefore $\tilde{\delta}(z_\alpha, \lambda)(f) = \lambda(\tilde{f}(z_\alpha)) \to \lambda(\tilde{f}(z)) = \phi(f)$ for every $f \in \mathcal{H}_{bc}(E,F)$. $\square$

Theorem 4.2. *Let π be as in Theorem* 2.1.

(1) *π is a continuous mapping from $\mathcal{M}(\mathcal{H}_{bc}(E,F))$ onto $E''_{w^*} \times \mathcal{M}(F)$ (where $E''_{w^*} = (E'', w^*)$).*

(2) *π defines a homeomorphism from $\{h \in \mathcal{M}(\mathcal{H}_{bc}(E,F)) : |h(f)| \leq \|f\|_{nB_E} \forall f \in \mathcal{H}_{bc}(E,F)\}$ onto $(n\overline{B}_{E''}; w^*) \times \mathcal{M}(F)\ \forall n \in IN$.*

Proof. (1) Given $(\phi_\alpha)_{\alpha \in I} \subset \mathcal{M}(\mathcal{H}_{bc}(E,F))$ such that $\phi_\alpha \xrightarrow{\tau_G} \phi$ in $\mathcal{M}(\mathcal{H}_{bc}(E,F))$, we have $\phi_\alpha(f \otimes 1_F) \to \phi(f \otimes 1_F)$ for all $f \in E'$ and $\phi_\alpha(1 \otimes b) \to \phi(1 \otimes b)$ for all $b \in F$. Consequently π is continuous.

(2) Let $A_n = \{h \in \mathcal{M}(\mathcal{H}_{bc}(E,F)) : |h(f)| \leq \|f\|_{nB_E}\ \forall f \in \mathcal{H}_{bc}(E,F)\}$. Given $(h_\alpha)_{\alpha \in I} \subset A_n$ such that $h_\alpha \xrightarrow{\tau_G} h$ in $\mathcal{M}(\mathcal{H}_{bc}(E,F))$, we have

$$|h(f)| \leq |h(f) - h_\alpha(f)| + |h_\alpha(f)| \leq |h(f) - h_\alpha(f)| + \|f\|_{nB_E}$$

for all $\alpha \in I$ and consequently $|h(f)| \leq \|f\|_{nB_E}$ for all $f \in \mathcal{H}_{bc}(E,F)$. So, A_n is closed in $\mathcal{M}(\mathcal{H}_{bc}(E,F))$.

Let $U_n = \{f \in \mathcal{H}_{bc}(E,F) : \|f\|_{nB_E} \leq 1\}$. Given $h \in A_n$, we have $|h(f)| \leq \|f\|_{nB_E}$ for all $f \in \mathcal{H}_{bc}(E,F)$ and consequently $|h(f)| \leq 1$ for all $f \in U_n$. So, $A_n \subset U_n^0 = \{T \in \mathcal{H}_{bc}(E,F)' : |T(f)| \leq 1 \forall f \in U_n\}$. Since U_n is a τ_b-neighbourhood of zero in $\mathcal{H}_{bc}(E,F)$, we have that U_n^0 is $\sigma(\mathcal{H}_{bc}(E,F)', \mathcal{H}_{bc}(E,F))$-compact and so A_n is a compact subset of $\mathcal{M}(\mathcal{H}_{bc}(E,F))$. We have by Theorem 2.1 that $\tilde{\delta} : E'' \times \mathcal{M}(F) \to \mathcal{M}(\mathcal{H}_{bc}(E,F))$ defined as in Theorem 4.1 is a bijection from $E'' \times \mathcal{M}(F)$ onto $\mathcal{M}(\mathcal{H}_{bc}(E,F))$ and $\tilde{\delta}^{-1} = \pi$. If $\tilde{\delta}(n\overline{B}_{E''} \times \mathcal{M}(F)) = A_n$, then the image of every closed subset of $n\overline{B}_{E''} \times \mathcal{M}(F)$ is a closed subset of A_n and consequently $\tilde{\delta}|_{n\overline{B}_{E''} \times \mathcal{M}(F)}$ is continuous. Now, for every $(z_0, \varphi) \in n\overline{B}_{E''} \times \mathcal{M}(F)$ we have

$$|(\delta_{z_0} \otimes \varphi)(f)| = |\varphi(\tilde{f}(z_0))| \leq \|\varphi \circ \tilde{f}\|_{nB_{E''}} = \|\varphi \circ \tilde{f}\|_{nB_E} \leq \|\varphi\| \, \|f\|_{nB_E} = \|f\|_{nB_E}$$

for all $f \in \mathcal{H}_{bc}(E,F)$ and so $\tilde{\delta}(n\overline{B}_{E''} \times \mathcal{M}(F)) \subset A_n$. Moreover given $h \in A_n$ there exists a unique $(z_0, \varphi) \in E'' \times \mathcal{M}(F)$ such that $h = \delta_{z_0} \otimes \varphi$. For every $\alpha \in E'$, $\varphi \circ (\alpha \otimes 1_F) \in E'$ and we have $\varphi \circ (\alpha \otimes 1_F)(x) = \varphi(\alpha(x) \cdot 1_F) = \alpha(x) \cdot \varphi(1_F) = \alpha(x)$ for all $x \in E$. As the mapping $z \in E'' \to \hat{\alpha}(z) = z(\alpha)$ is a w^*-continuous extension of $\varphi \circ (\alpha \otimes 1_F) = \alpha$ to E'', we have $\varphi \circ \widetilde{(\alpha \otimes 1_F)}(z) = \hat{\alpha}(z)$ for all $z \in E''$ and so $|\delta_{z_0} \otimes \varphi(\alpha \otimes 1_F)| = |\varphi \circ \widetilde{(\alpha \otimes 1_F)}(z_0)| = |\hat{\alpha}(z_0)|$. This implies

$$\|z_0\| = \sup_{\substack{\alpha \in E' \\ \|\alpha\| \leq 1}} |\hat{\alpha}(z_0)| = \sup_{\substack{\alpha \in E' \\ \|\alpha\| \leq 1}} |(\delta_{z_0} \otimes \varphi)(\alpha \otimes 1_F)| \leq \sup_{\substack{\alpha \in E' \\ \|\alpha\| \leq 1}} \|\alpha \otimes 1_F\|_{nB_E} \leq n.$$

Consequently, $A_n \subset \tilde{\delta}(n\overline{B}_{E''} \times \mathcal{M}(F))$ and this completes the proof. □

Remark. (a) The mapping $\tilde{\delta}$ is continuous from $E'' \times \mathcal{M}(F))$ onto $\mathcal{M}(\mathcal{H}_{bc}(E,F))$ for all F, but it is not continuous from $(E'', w^*) \times \mathcal{M}(F)$ onto $\mathcal{M}(\mathcal{H}_{bc}(E,F))$, even if $F = \mathbb{C}$.

(b) A slight modification of arguments from Theorem 4.2 shows that the mapping π_1 defined in Proposition 3.1 is also continuous from $\mathcal{M}(\mathcal{H}_{bK}(E,F))$ onto $\mathcal{M}(\mathcal{H}_b(E)) \times \mathcal{M}(F)$.

5 THE ALGEBRA $\mathcal{H}_b(E,F)$

When we work with $\mathcal{H}_b(E,F)$, we can't use Theorem A. In this case, we will use the Aron–Berner extension for elements of $\mathcal{H}_b(E)$. Davie and Gamelin presented in [11] a nice construction of this extension whose idea is the following: by using the w^*-density of E in E'', every continuous n-linear mapping $A : E \times \cdots \times E \to \mathbb{C}$ can be extended to a continuous n-linear mapping $\hat{A} : E'' \times E'' \times \cdots \times E'' \to \mathbb{C}$ beginning with the last variable and working backwards to the first. By using the Taylor representation of $f \in \mathcal{H}_b(E)$ at the origin we get

$$f(x) = \sum_{n=0}^{\infty} A_n(\overbrace{x, \cdots, x}^{n \text{ times}})$$

where $A_0 \in \mathbb{C}$ and, for each $n \geq 1$, A_n is a continuous n-linear mapping from E^n into $\mathbb{C}$. Now, for each $f \in \mathcal{H}_b(E)$, the mapping

$$\hat{f}(z) = \sum_{n=0}^{\infty} \hat{A}_n(\overbrace{z, \cdots, z}^{n \text{ times}})$$

is an element of $\mathcal{H}_b(E'')$ such that $\hat{f}|_E = f$. For details, we refer to [11].

To each $(z, \varphi) \in E'' \times \mathcal{M}(F)$ we can associate a mapping $\delta_z \diamond \varphi : \mathcal{H}_b(E,F) \to \mathbb{C}$ defined by $\delta_z \diamond \varphi(f) := \widehat{\varphi \circ f}(z)$ for all $f \in \mathcal{H}_b(E,F)$, where $\widehat{\varphi \circ f}$ denotes the Aron–Berner extension of $\varphi \circ f$. It is easy to verify that $\delta_z \diamond \varphi \in \mathcal{M}(\mathcal{H}_b(E,F))$.

Proposition 5.1. *Let $\hat{\delta} : E'' \times \mathcal{M}(F) \to \mathcal{M}(\mathcal{H}_b(E,F))$ be defined by $\hat{\delta}(z,\varphi)(f) = \delta_z \diamond \varphi(f)$. Then, the following results hold:*

(1) $\hat{\delta}$ *is injective.*

(2) *If E' has the approximation property and $\mathcal{P}_f(E,F)$ is dense in $\mathcal{P}(E,F)$, then $\delta_z \diamond \varphi \equiv \delta_z \otimes \varphi$ for every $(z,\varphi) \in E'' \times \mathcal{M}(F)$. Consequently, $\hat{\delta}$ is a bijection.*

(3) *If $\hat{\delta}(E'' \times \mathcal{M}(F)) = \mathcal{M}(\mathcal{H}_b(E,F))$, then $\hat{\delta}^{-1}$ is a homeomorphism from $\{h \in \mathcal{M}(\mathcal{H}_b(E,F)) : |h(f)| \leq \|f\|_{nB_E} \forall f \in \mathcal{H}_b(E,F)\}$ onto $(n\overline{B}_{E''}, w^*) \times \mathcal{M}(F)$ for all $n \in IN$.*

Proof. (1) If $(z_1,\varphi_1),(z_2,\varphi_2) \in E'' \times \mathcal{M}(F)$ is such that $\hat{\delta}(z_1,\varphi_1) = \hat{\delta}(z_2,\varphi_2)$ we have $\delta_{z_1} \diamond \varphi_1(f) = \delta_{z_2} \diamond \varphi_2(f)$ for all $f \in \mathcal{H}_b(E,F)$ and so $\varphi_1(b) = \delta_{z_1} \diamond \varphi_1(1 \otimes b) = \delta_{z_2} \diamond \varphi_2(1 \otimes b) = \varphi_2(b)$ for all $b \in F$. Now, given $x \in E$ and $T \in E'$ arbitrary, we have $\varphi_1 \circ (T \otimes 1_F)(x) = \varphi_1(T(x) \cdot 1_F) = T(x)$ and consequently $\delta_z \diamond \varphi_1(T \otimes 1_F) = \hat{T}(z)$ for all $z \in E''$. Since $\varphi_1 = \varphi_2$ and

$$z_1(T) = \hat{T}(z_1) = \delta_{z_1} \diamond \varphi_1(T \otimes 1_F) = \delta_{z_2} \diamond \varphi_1(T \otimes 1_F) = \hat{T}(z_2) = z_2(T)$$

for every $T \in E'$ we conclude that $z_1 = z_2$.

(2) For every $\alpha \in E'$ and $b \in F$ we have $\tilde{\alpha}^n(z) \cdot \varphi(b) = \hat{\alpha}^n(z) \cdot \varphi(b)$ for all $z \in E''$ and $\varphi \in \mathcal{M}(F)$. Since $\tilde{\alpha}^n(\cdot) \cdot \varphi(b)$ and $\hat{\alpha}^n(\cdot) \cdot \varphi(b)$ are both w^*-continuous extensions of $\alpha^n(\cdot) \cdot \varphi(b)$ to E'', we get

$$\delta_z \otimes \varphi(\alpha^n \otimes b) = \widetilde{\varphi \circ \alpha^n \otimes b}(z) = \tilde{\alpha}^n(z) \cdot \varphi(b) = \hat{\alpha}(z) \cdot \varphi(b) = \delta_z \diamond \varphi(\alpha^n \otimes b).$$

Now it is enough to use the continuity of $\delta_z \diamond \varphi$ and $\delta_z \otimes \varphi$ in $\mathcal{H}_b(E,F)$ and to remember that, under our conditions, $\mathcal{P}_f(E,F)$ is dense in $\mathcal{H}_b(E,F)$.

(3) A (slight) modification of arguments from Theorem 4.2 gives the proof. □

Theorem 5.2. *Let E be a Banach space such that E' has the approximation property and let F be a uniform Banach algebra with identity. Then $\hat{\delta}$ defined as in Proposition 5.1 defines a bijection from $E'' \times \mathcal{M}(F)$ onto $\mathcal{M}(\mathcal{H}_b(E,F))$ if and only if $\mathcal{P}_f(E,F)$ is dense in $\mathcal{P}(E,F)$.*

Proof. Given $f \in \mathcal{P}(^kE,F)$, for each $n \in IN$ we consider the restriction of $\delta_f \circ \hat{\delta}$ to $n\overline{B}_{E''} \times \mathcal{M}(F)$ (where δ_f is the evaluation). By Proposition 5.1 $\hat{\delta}$ is a homeomorphism from $X_n := (n\overline{B}_{E''}, w^*) \times \mathcal{M}(F)$ onto $A_n \subset \mathcal{M}(\mathcal{H}_b(E,F))$ and consequently, $\delta_f \circ \hat{\delta}|_{n\overline{B}_{E''} \times \mathcal{M}(F)} \in \mathcal{C}(X_n)$, where X_n is a compact subset of $(E'', w^*) \times \mathcal{M}(F)$. It is well-known that $\mathcal{C}(X_n)$ is isometric to $\mathcal{C}((n\overline{B}_{E''}, w^*), \mathcal{C}(\mathcal{M}(F)))$ and so we can consider $\delta_f \circ \hat{\delta}$ as an element of $\mathcal{C}(E'', \mathcal{C}(\mathcal{M}(F)))$ that is w^*--$\|\ \|_{\mathcal{M}(F)}$-continuous on bounded subsets of E''.

Since F is a uniform algebra it is well-known that F is isomorphically isometric to $\hat{F}$, where $\hat{F} = \{\hat{b} : b \in F\}$ and $\hat{b}$ is the Gelfand transform of b (cf. [14]). So for every $x \in E$, $f(x) \in \hat{F}$. This means that for all $\varphi \in \mathcal{M}(F)$ we have $f(x)(\varphi) = \varphi(f(x))$ and consequently, for every $x \in E$ and for every $\varphi \in \mathcal{M}(F)$ we have

$$(\delta_f \circ \hat{\delta}(x))(\varphi) = (\delta_f \circ \hat{\delta}(x,\cdot))(\varphi) = \widetilde{\varphi \circ f}(x) = \varphi \circ f(x) = f(x)(\varphi).$$

Therefore $f(x) = \delta_f \circ \hat{\delta}(x)$ for every $x \in E$ and so $f \in \mathcal{P}_{wu}(^kE,F)$. Since

$\mathcal{P}_{wu}({}^kE, F) = \overline{\mathcal{P}_f({}^kE, F)}$ whenever E' has the approximation property it follows that $\mathcal{P}_f({}^kE, F)$ is dense in $\mathcal{P}({}^kE, F)$. As this is true for every $k \in IN$, we have the density of $\mathcal{P}_f(E, F)$ in $\mathcal{P}(E, F)$.

The converse follows by Corollary 2.3 and the proof is complete. □

Remark. If F is a W^*-algebra (i.e., a C^*-algebra that is a dual of a Banach space) we can prove an analogous of Theorem 5.2 where $\tilde{\delta}$ appears instead of $\hat{\delta}$.

Proposition 5.3. *Suppose that E' has the approximation property, $\mathcal{P}_f(E, F)$ is dense in $\mathcal{P}(E, F)$ and $\hat{\delta} : E'' \times \mathcal{M}(F) \to \mathcal{M}(\mathcal{H}_b(E, F))$ is defined as in Proposition 5.1. Then $\hat{\delta}(E \times \mathcal{M}(F))$ is dense in $\mathcal{M}(\mathcal{H}_b(E, F))$.*

Proof. The proposition follows from Theorem 4.1 since, under the hypothesis we have $\mathcal{H}_b(E, F) = \mathcal{H}_{bc}(E, F)$ and $\hat{\delta} = \tilde{\delta}$. □

Remark. By Corollary 4.7 of [5] we can also prove that $E \times \mathcal{M}(F)$ is $\mathcal{P}(E) \otimes F$-dense in $\mathcal{M}(\mathcal{H}_b(E, F))$.

6 THE ALGEBRA $\mathcal{A}_{wu}(B_E; F)$

Let $\mathcal{A}_{wu}(nB_E; F)$ be the space of all mappings $f : n\overline{B}_E \to F$ that are holomorphic on nB_E and uniformly weakly continuous on $n\overline{B}_E := \{x \in E : \|x\| \leq n\}$ endowed with the topology generated by the norm defined by $\|f\|_n := \sup_{x \in nB_E} \|f(x)\|$ for every $f \in \mathcal{A}_{wu}(nB_E; F)$. It is easy to verify that $\mathcal{A}_{wu}(nB_E; F)$ is a Banach algebra.

Proposition 6.1. *The space of all $f \in \mathcal{A}_{wu}(nB_E; F)$ such that $f = g|_{nB_E}$ for some $g \in \mathcal{H}_{wu}(E, F)$ is dense in $\mathcal{A}_{wu}(nB_E; F)$.*

Proof. Without loss of generality we may suppose $f \in \mathcal{A}_{wu}(B_E; F)$. Given $\epsilon > 0$ there exists $\delta > 0$ and $\varphi_1, \varphi_2, \ldots, \varphi_k \in E'$, so that $\|f(x) - f(y)\| < \epsilon$ whenever $x, y \in B_E$ and satisfy $|\varphi_i(x - y)| < \delta$ for all $i = 1, \ldots, k$. Take $M \geq \|\varphi_i\|$ for all $i = 1, \ldots, k$, and $\delta > 0$ sufficiently small so that $1 - \frac{\delta}{M} > 0$. Let $s \in IR$ such that $1 - \frac{\delta}{M} < \frac{1}{s} < 1$. Define $f_s(x) := f(\frac{1}{s}x)$ for all $x \in B_E$. It is easy to show that $\|f - f_s\|_1 < \epsilon$ and $f_s \in \mathcal{A}_{wu}(sB_E; F)$. Now let $f = \sum_{j=0}^{\infty} P_j$ be the Taylor representation of f at zero. There exists $N > 0$ such that $\|f_s - \sum_{j=0}^{N} s^{-j} P_j\|_1 < \epsilon$ and, consequently, $\|f - \sum_{j=0}^{N} s^{-j} P_j\|_1 < 2\epsilon$. Since $P_j \in \mathcal{P}_{wu}({}^jE, F)$ for every $j \in IN$ we have the proof. □

For each $n \in IN$ let $j_n : \mathcal{A}_{wu}(nB_E; F)' \to \mathcal{H}_{wu}(E, F)'$ be defined by $j_n(\varphi)(f) := \varphi(f|_{nB_E})$ for all $\varphi \in \mathcal{A}_{wu}(nB_E; F)'$ and for all $f \in \mathcal{H}_{wu}(E; F)$. Let $\pi_n : \mathcal{H}_{wu}(E, F) \to \mathcal{A}_{wu}(nB_E; F)$ be defined by $\pi_n(f) := f|_{nB_E}$ for all $f \in \mathcal{H}_{wu}(E, F)$. It is clear that $j_n(\varphi) = \varphi \circ \pi_n \in \mathcal{H}_{wu}(E, F)'$ and that j_n is a linear continuous mapping. We remark that j_n is injective. Indeed, if $\varphi_1, \varphi_2 \in \mathcal{A}_{wu}(nB_E; F)'$ are such that $j_n(\varphi_1) = j_n(\varphi_2)$ we have $\varphi_1(f|_{nB_E}) = \varphi_2(f|_{nB_E})$ for all $f \in \mathcal{H}_{wu}(E, F)$. By using the density of $\pi_n(\mathcal{H}_{wu}(E,F))$ in $\mathcal{A}_{wu}(nB_E;F)$ and the continuity of φ_1 and φ_2 we get $\varphi_1=\varphi_2$ in $\mathcal{A}_{wu}(nB_E;F)$.

Lemma 6.2. $\mathcal{H}_{wu}(E,F)'=\bigcup_{n\in IN} j_n(\mathcal{A}_{wu}(nB_E;F)')$.

Proof. Given $u\in\mathcal{H}_{wu}(E,F)'$ we want to find $n\in IN$ and $\bar{u}_n\in\mathcal{A}_{wu}(B_E;F)'$ such that $j_n(\bar{u}_n)=u$. As u is linear and continuous there exist $n\in IN$ and $c>0$ such that $\|u(f)\|\leq c\|f\|_{nB_E}=c\|f|_{nB_E}\|_n$ for every $f\in\mathcal{H}_{wu}(E,F)$. Let $u_n:\pi_n(\mathcal{H}_{wu}(E,F))\to\mathbb{C}$ be defined by $u_n(f):=u(g_f)$ for all $f\in\pi_n(\mathcal{H}_{wu}(E,F))$, where $g_f\in\mathcal{H}_{wu}(E,F)$ is such that $\pi_n(g_f)=f$. It is clear that u_n is a well defined linear mapping. As $|u_n(f)|=|u(g_f)|\leq c\|g_f\|_n=c\|f\|_n$ for all $f\in\pi_n(\mathcal{H}_{wu}(E,F))$, $u_n\in(\pi_n(\mathcal{H}_{wu}(E,F)))'$ and since $\pi_n(\mathcal{H}_{wu}(E,F))$ is dense in $\mathcal{A}_{wu}(nB_E;F)$, there exists $\bar{u}_n\in\mathcal{A}_{wu}(nB_E;F)'$ such that $\bar{u}_n|_{\pi_n(\mathcal{H}_{wu}(E,F))}=u_n$. As $j_n(\bar{u}_n)(g)=\bar{u}_n(\pi_n(g))=u_n(\pi_n(g))=u(g)$ for every $g\in\mathcal{H}_{wu}(E,F)$ we have $j_n(\bar{u}_n)=u$. Since the other inclusion is trivial, the proof is complete. □

Note that in Proposition 6.1 and Lemma 6.2 it is enough to consider F as a Banach space.

Proposition 6.3. $\mathcal{M}(\mathcal{H}_{wu}(E,F))=\bigcup_{n\in IN} j_n[\mathcal{M}(\mathcal{A}_{wu}(nB_E;F))]$.

Proof. Suppose first that $u=j_n(\varphi)$, where $\varphi\in\mathcal{M}(\mathcal{A}_{wu}(nB_E;F))$ (n arbitrary). By Lemma 6.2, $u\in\mathcal{H}_{wu}(E,F)'$. It is easy to verify that $u(f\cdot g)=u(f)\cdot u(g)$ for all $f,g\in\mathcal{H}_{wu}(E,F)$. Since j_n is injective we have $u\neq 0$ and so $u\in\mathcal{M}(\mathcal{H}_{wu}(E,F))$.

Conversely, assume that $u\in\mathcal{M}(\mathcal{H}_{wu}(E,F))$. Let $\bar{u}_n\in\mathcal{A}_{wu}(nB_E;F)'$ be such that $j_n(\bar{u}_n)=u$ defined as in the proof of Lemma 6.2. Since $\bar{u}_n$ is continuous, by Proposition 6.1 we get $\bar{u}_n(f\cdot g)=\bar{u}_n(f)\cdot\bar{u}_n(g)$ for all $f,g\in\mathcal{A}_{wu}(nB_E;F)$. Therefore $\bar{u}_n\in\mathcal{M}(\mathcal{A}_{wu}(nB_E;F))$ since $\overline{u_n}\neq 0$ by the linearity of j_n. □

Theorem 6.4. *If E' has the approximation property, $\mathcal{M}(\mathcal{A}_{wu}(B_E;F))$ is homeomorphic to $(\overline{B}_{E''};w^*)\times\mathcal{M}(F)$.*

Proof. Given $f\in\mathcal{A}_{wu}(B_E;F)$, since f is uniformly w-continuous on B_E, $\sigma(E'',E')|_E=\sigma(E,E')$ and $\overline{B}_{E''}=\overline{B}_E^{w^*}$, there exists a unique extension of f to $\overline{B}_{E''}$ which is w^*-continuous. It is easy to check that for every $f\in\mathcal{A}_{wu}(B_E;F)$ such that there exists $g\in\mathcal{H}_{wu}(E,F)$ satisfying $g|_{B_E}=f$, $\tilde{g}|_{\overline{B}_{E''}}$ is the unique w^*-continuous extension of f to $\overline{B}_{E''}$. So, it is natural to denote by $\tilde{f}$ the unique w^*-continuous extension of $f\in\mathcal{A}_{wu}(B_E;F)$ to $\overline{B}_{E''}$.

Now, for every $z\in\overline{B}_{E''}$ we define $\alpha_z:\mathcal{A}_{wu}(B_E;F)\to F$ by $\alpha_z(f):=\tilde{f}(z)$ for all $f\in\mathcal{A}_{wu}(B_E;F)$. We recall that $\delta_z:\mathcal{H}_{wu}(E,F)\to F$ was defined by $\delta_z(f):=\tilde{f}(z)$ for all $f\in\mathcal{H}_{wu}(E,F)$. Clearly $\delta_z(f)=\alpha_z(f|_{B_E})$.

Let $\tilde{\alpha}:\overline{B}_{E''}\times\mathcal{M}(F)\to\mathcal{M}(\mathcal{A}_{wu}(B_E;F))$ be defined by $\tilde{\alpha}(z,\lambda):=\alpha_z\otimes\lambda$ for all $(z,\lambda)\in\overline{B}_{E''}\times\mathcal{M}(F)$, where $\alpha_z\otimes\lambda(f):=\lambda(\alpha_z(f))\in\mathbb{C}$ for every $f\in$

$\mathcal{A}_{wu}(B_E; F)$. A slight modification of arguments used in the proof of Theorem 2.1(2) shows that $\alpha_z \otimes \lambda \in \mathcal{M}(\mathcal{A}_{wu}(B_E; F))$.

Let $(z_1, \lambda_1), (z_2, \lambda_2) \in B_{E''} \times \mathcal{M}(F)$ such that $(z_1, \lambda_1) \neq (z_2, \lambda_2)$. If $z_1 \neq z_2$ there exists $T \in E'$ such that $z_1(T) \neq z_2(T)$. Then $T|_{B_E} \otimes 1_F \in \mathcal{A}_{wu}(B_E; F)$ is such that

$$\begin{aligned}\tilde{\alpha}(z_1, \lambda_1)(T|_{B_E} \otimes 1_F) = \lambda_1(\tilde{T}(z_1) \cdot 1_F) = \tilde{T}(z_1) = z_1(T) \neq z_2(T) \\ = \tilde{\alpha}(z_2, \lambda_2)(T|_{B_E} \otimes 1_F).\end{aligned}$$

Now, if $z_1 = z_2$ we must have $\lambda_1 \neq \lambda_2$ and consequently there exists $b \in F$ such that $\lambda_1(b) \neq \lambda_2(b)$. Then $1 \otimes b \in \mathcal{A}_{wu}(B_E; F)$ is such that

$$\tilde{\alpha}(z_1, \lambda_1)(1 \otimes b) = \lambda_1(b) \neq \lambda_2(b) = \tilde{\alpha}(z_2, \lambda_2)(1 \otimes b).$$

Hence $\tilde{\alpha}(z_1, \lambda_1) \neq \tilde{\alpha}(z_2, \lambda_2)$ whenever $(z_1, \lambda_1) \neq (z_2, \lambda_2)$. Let us show that $\tilde{\alpha}$ is onto. Given $\phi \in \mathcal{M}(\mathcal{A}_{wu}(B_E; F))$ we have $j_1(\phi) \in \mathcal{M}(\mathcal{H}_{wu}(E, F))$ by Proposition 6.3. Since E' has the approximation property, in view of Corollary 2.2 there exists $(z, \lambda) \in E'' \times \mathcal{M}(F)$ such that $j_1(\phi) = \delta_z \otimes \lambda$ i.e., $j_1(\phi)(f) = \lambda(\tilde{f}(z))$ for all $f \in \mathcal{H}_{wu}(E, F)$. We claim that $z \in \overline{B}_{E''}$. Indeed, since

$$|(\delta_z \otimes \lambda)(f)| = |\lambda(\tilde{f}(z))| = |j_1(\phi)(f)| = |\phi(f|_{B_E})| \leq \|\phi\| \, \|f\|_1 = \|f\|_1$$

for all $f \in \mathcal{H}_{wu}(E, F)$ we get

$$\|z\| = \sup_{\substack{T \in E' \\ \|T\| \leq 1}} |z(T)| = \sup_{\substack{T \in E' \\ \|T\| \leq 1}} |\tilde{T}(z)| = \sup_{\substack{T \in E' \\ \|T\| \leq 1}} |(\delta_z \otimes \lambda)(T \otimes 1_F)| \leq \sup_{\substack{T \in E' \\ \|T\| \leq 1}} \|T \otimes 1_F\|_1 \leq 1.$$

As ϕ and α_z are continuous on $\mathcal{A}_{wu}(B_E; F)$ and by Proposition 6.1, $\{f|_{B_E} : f \in \mathcal{H}_{wu}(E, F)\}$ is dense in $\mathcal{A}_{wu}(B_E; F)$ we have $\alpha_z \otimes \lambda = \phi$ since $(\alpha_z \otimes \lambda)(f|_{B_E}) = \lambda(\tilde{f}(z)) = (\delta_z \otimes \lambda)(f) = j_1(\phi)(f) = \phi(f|_{B_E})$ for all $f \in \mathcal{H}_{wu}(E, F)$. Therefore, $\tilde{\alpha}$ is onto. It remains to show that $\tilde{\alpha}$ is a homeomorphism. Let $\pi : \mathcal{M}(\mathcal{A}_{wu}(B_E; F)) \to (\overline{B}_{E''}, w^*) \times \mathcal{M}(F)$ be defined by $\pi(\phi) := (\varphi|_{E'}, \psi)$, where $\varphi(f) := \phi(f \otimes 1_F)$ for every $f \in E'$ and $\psi(b) := \phi(1 \otimes b)$ for every $b \in F$. It is not hard to verify that π is well defined and $\pi \equiv \tilde{\alpha}^{-1}$. Now, consider a net (ϕ_β) such that ϕ_β converges to ϕ in $\mathcal{M}(\mathcal{A}_{wu}(B_E; F))$ i.e., $\phi_\beta(f) \to \phi(f)$ for every $f \in \mathcal{A}_{wu}(B_E; F)$. For all $f \in E'$ we have $\varphi_\beta(f) := \phi_\beta(f \otimes 1_F) \to \phi(f \otimes 1_F) := \varphi(f)$ and $\psi_\beta(b) := \phi_\beta(1 \otimes b) \to \phi(1 \otimes b) := \psi(b)$ for all $b \in F$. From this we conclude that $\pi(\phi_\beta)$ converges to $\pi(\phi)$ and hence π is continuous.

Finally, since $\tilde{\alpha}$ is an injective mapping from $(\overline{B}_{E''}, w^*) \times \mathcal{M}(F)$ onto the compact space $\mathcal{M}(\mathcal{A}_{wu}(B_E; F))$ whose inverse π is continuous, a compactness argument shows that $\tilde{\alpha}$ is a homeomorphism. $\square$

It is interesting to observe that similar arguments to those used in Theorem 2.1 provide a different proof of the existence of a bijection between $\overline{B}_{E''} \times \mathcal{M}(F)$ and $\mathcal{M}(\mathcal{A}_{wu}(B_E; F))$.

Moreover, similar arguments to those used in Theorem 4.1 prove the following:

Corollary 6.5. *If E' has the approximation property, then $\mathcal{M}(\mathcal{A}_{wu}(B_E; F))$ contains an isomorphic copy of $(B_E, w) \times \mathcal{M}(F)$ that is dense in $\mathcal{M}(\mathcal{A}_{wu}(B_E; F))$.*

The same arguments used in Theorem 6.4 show that $\mathcal{M}(\mathcal{A}(nB_E; F))$ is homeomorphic to $(nB_{E''}, w^*) \times \mathcal{M}(F)$, if E' has the approximation property.

REFERENCES

1. Alencar, R., R.M. Aron and S. Dineen – A reflexive space of holomorphic functions in infinitely many variables. Proc AMS **90** (3), 407–411 (1984).
2. Alencar, R. and K. Floret – Weak continuity of multilinear mappings on Tsirelson's space. Quaestiones Mathematicae **21**, 177–186 (1998).
3. Aron, R.M. – Weakly uniformly continuous and weakly sequentially continuous entire functions, in: Advances in Holomorphy (Ed. J A. Barroso). Math. Studies **34**, North Holland, Amsterdam, 47–66 (1979)
4. Aron, R.M. – Compact polynomials and compact differentiable mappings between Banach spaces. Seminaire P. Lelong (Analyse) 1974/75 (Ed. P. Lelong). LNM, **524**, Springer Verlag, Berlin, 213–222 (1976).
5. Aron, R.M., B.J Cole and T.W Gamelin – Spectra of algebras of analytic functions on a Banach space J. Reine Angew Math. **415**, 51–93 (1991).
6. Aron, R.M., B.J. Cole and T.W. Gamelin – Weak-star continuous analytic functions. Can. J. of Math. **47** (4), 673–683 (1995).
7. Aron, R.M. and S. Dineen – Q-reflexive Banach spaces. Rocky Mount J. Math **27** (4), 1009–1025 (1997)
8. Aron, R.M., C. Hervés and M Valdivia – Weakly continuous mappings on Banach spaces J Funct. Anal. **52**, 189–204 (1983).
9. Aron, R.M. and J.B. Prolla – Polynomial approximation of differentiable functions on Banach spaces J. Reine Angew. Math. **313**, 195–216 (1980).
10. Aron, R M. and M. Schottenloher – Compact holomorphic mappings on Banach spaces and the approximation property. J Funct. Anal. **21** (1), 7–30 (1976).
11. Davie, A.M. and T W. Gamelin – A theorem on polynomial star approximation. Proc Amer Math. Soc. **106**, 351–356 (1989).
12. Dineen, S. – Complex Analysis in Locally Convex Spaces Math. Studies **57**, North Holland, Amsterdam (1981).
13. Dineen, S. – Entire functions on c_0. J. Funct. Analysis **52**, 205–218 (1983).
14. Gamelin, T.W – Uniform Algebras Second Edition, New York (1984)
15. Gupta, C.P – Malgrange theorem for nuclearly entire functions of bounded type on a Banach space Notas de Matemática **37**, IMPA, Rio de Janeiro (1968).
16. Gutierrez, J – Weakly continuous functions on Banach spaces not containing l_1. Proc. Amer. Math. Soc **119**, 147–152 (1993)
17. Hoffman, K. – Banach spaces of analytic functions. Second Edition. Dove (1989).
18. Moraes, L A. – Extension of holomorphic mappings from E to E''. Proceedings of the A M.S. **118** (2), 455–461 (1993).
19. Moraes, L.A Weakly continuous holomorphic mappings. Proc of the Royal Irish Acad. **97** A (2), 139–144 (1997).
20. Mujica, J – Complex Analysis in Banach Spaces. Math. Studies **120**, North Holland, Amsterdam (1986)
21. Pelczynski, A – A theorem of Dunford-Pettis type for polynomial operators. Bull. Acad Pol. Science **XI** (6), 379–386 (1963).
22. Tonev, T.V. – Big-planes boundaries and function algebras. Math. Studies **172**, North Holland, Amsterdam (1992)
23. Tsirelson, B S. – Not every Banach space contains an embedding of l_p or c_0. Functional Anal. Appl. **8**, 138–141 (1974).